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Synthesis of Time-Optimal Control for Linear Processes

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INTRODUCTION

In this paper a method of synthesis of time-optimal control for linear processes is developed. The development parallels and generalizes that of Neustadt [1]. The motivation for this generalization is provided by the results presented in [2] and [3].

The problem to be considered here is the following: Suppose the state of a physical system is described by an n-dimensional vector $x(t)$ where the equations of motion are of the form

$$\dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \tag{1}$$

in which $u(t)$ denotes an r-dimensional vector (the control), $A(t)$ is an $n \times n$ matrix and $B(t)$ is an $n \times r$ matrix. The state of the system is given at $t = 0$ as $x(0) = x^0$. A target set G is given where it is assumed that G is a constant, compact, convex nonempty subset of R^n. The control $u(t)$ is to satisfy the constraint $|u_i(t)| \leqslant 1$ for $t > 0$ and $i = 1, 2, \cdots, r$. Furthermore the system (1) is assumed to be "normal." Then it is desired to find a control $u(t)$ defined on an interval $(0, t_1)$ such that the corresponding solution of (1) satisfies $x(t_1) \in G$ where t_1 is a minimum. Further restrictions will be placed on G later in the development.

It is known that an optimal control for the above problem must be an extremal control, where an extremal control is one that satisfies the following conditions. Let $\Phi(t)$ be the fundamental matrix for $\dot{x} = Ax$ with $\Phi(0) = I$. Then $u(t)$ is extremal if

(i) There exists an n-vector η with $\|\eta\| = 1$ such that $u(t) = u(\eta, t)$ where

$$u_i(\eta, t) = \text{sign}\left[\sum_{j,k=1}^{n} \eta_j \psi_{kj}(t)\, b_{ki}(t)\right] \qquad i = 1, 2, \cdots, r,$$

where

$$\Psi(t) = [\Phi^{-1}(t)]', \qquad 0 \leqslant t \leqslant T.$$

(ii) $\Psi(T)\,\eta$ is normal to a support of G at $x(T)$ and is directed into the halfspace containing G ($x(T) \in \partial G$).

(iii) $$\Psi(T)\,\eta \cdot [A(T)\,x(T) + B(T)\,u(\eta, T)] \geqslant 0.$$

Thus, it is desired to find a solution to the following equation

$$x(T) = \Phi(T)\,x^0 + \Phi(T)\int_0^T \Phi^{-1}(s)\,B(s)\,u(\eta, s)\,ds \tag{2}$$

with $T > 0$, $x(T)$ a boundary point of G such that (ii) and (iii) are satisfied.

Development of Synthesis Method

The vector $\Psi(t)\,\eta$ will, for simplicity, be denoted in the following by $\zeta(t)$. It will also be assumed that $\| \eta \| = 1$ in all that follows. With these conventions let $w(\eta, t)$ denote a point of the boundary of G such that $\zeta(t)$ is normal to a support of G at $w(\eta, t)$ and is directed into the halfspace containing G and the scalar product $\zeta(t) \cdot [A(t)\,w(\eta, t) + B(t)\,u(\eta, t)]$ is nonnegative. It is possible that $w(\eta, t)$ may not be defined for all η and all $t > 0$. For any η and any $t > 0$ it is possible to find a point $v(\eta, t)$ on the boundary of G such that $\zeta(t)$ is normal to a support of G at $v(\eta, t)$ and $\zeta(t)$ is directed into the halfspace containing G since G is compact and convex. But it may be that for some η and some $t > 0$ the scalar product

$$\zeta(t) \cdot [A(t)\,v(\eta, t) + B(t)\,u(\eta, t)] < 0$$

for every possible $v(\eta, t)$. A simple example illustrates this phenomenon. Consider the system of the form (1) with $A(t)$ and $B(t)$ given by

$$A(t) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B(t) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

If $G = \{(1, 1)\}$, then

$$v(\eta, t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for any η and any $t > 0$. But

$$\zeta(t) \cdot [A(t)\,v(\eta, t) + B(t)\,u(\eta, t)] < 0$$

for any η and $t > 0$ such that

$$2\zeta_1(t) < -\sqrt{2}\,\sqrt{[\zeta_1(t)]^2 + [\zeta_2(t)]^2}.$$

This is readily verified by considering the possible trajectories of the system passing through the point (1, 1) in the phase plane.

Also, it should be noted that $w(\eta, t)$ may not be single-valued. For example, consider the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}1\\ 1\end{bmatrix} u, \qquad |u| \leqslant 1,$$

$$G = \{(x_1, x_2) : x_1 = 0, |x_2| \leqslant 1\}.$$

It is clear from considering the possible trajectories passing through points of G that for any η and $t > 0$ for which $\zeta_2(t) = 0$, then $w(\eta, t) = G$.

One favorable thing to be said for $w(\eta, t)$, however, is that $\eta \cdot w(\eta, 0)$ is continuous and single-valued wherever it is defined.

Now the equation

$$w(\eta, T) = \Phi(T)\, x^0 + \Phi(T) \int_0^T \Phi^{-1}(s)\, B(s)\, u(\eta, s)\, ds \tag{3}$$

is equivalent to Eq. (2) with the restrictions given for it.

Equation (3) may be premultiplied by $\Phi^{-1}(t)$ to obtain

$$x^0 - \Phi^{-1}(T)\, w(\eta, T) + \int_0^T \Phi^{-1}(s)\, B(s)\, u(\eta, s)\, ds = 0 \tag{4}$$

for which a solution is desired.

Let

$$z(t, \eta) = \int_0^t \Phi^{-1}(s)\, B(s)\, u(\eta, s)\, ds.$$

Then the point $z(t, \eta)$ in Euclidean n-space is a boundary point of the set $C(t)$, where $C(t)$ is the set of all initial conditions $x(0)$ from which the origin can be reached in the interval $[0, t]$. It is known [1] that $C(t)$ is compact, convex, and grows continuously with t. Furthermore, the boundary of $C(t)$ is the set of all $z(\tau, \eta)$ with $\tau = t$ and $z(t, \eta)$ is the vector in $C(t)$ which has a maximum projection in the direction of η. For "normal" systems there is a unique $z(t, \eta)$ for each η but the converse is not necessarily true, i.e., $C(t)$ is strictly convex but the boundary is not necessarily smooth.

The algorithm to be obtained will be such that an initial η will be chosen and an equation of motion for η with respect to a parameter $\tau \geqslant 0$ will be found such that the positive limit set is the set of η's for which Eq. (4) has a solution.

Since, for a given value of η, Eq. (4) may not have a solution a scalar equation which is related to (4) will be obtained which will have a solution for any value of η if G satisfies condition (A) given below. This equation is

$$f(t, \eta; x^0) = 0 \tag{5}$$

where

$$f(t, \eta; x^0) = \eta \cdot [x_0 - \Phi^{-1}(t)\, w(\eta, t) + z(t, \eta)]. \tag{6}$$

In (6) and all that follows $w(\eta, t)$ is to be modified (without change in notation) so that it is single-valued by choosing an arbitrary point in the set $w(\eta, t)$ as previously defined as the value of $w(\eta, t)$.

Before proceeding with analytic descriptions of the properties of $f(t, \eta; x^0)$, a geometrical interpretation of (5) and (6) will be given. For any t and x^0 the set $C(t, x^0)$ can be defined as the translation of $C(t)$ by x^0. Then $x^0 + z(t, \eta)$ represents a point on the boundary of $C(t, x^0)$ which is a point of support corresponding to η, as shown in Fig. 1. Also illustrated in figure 1 are $\zeta(t_1)$ and $w(\eta, t_1)$. Furhtermore $\Phi^{-1}(t_1)\, w(\eta, t_1)$ is shown as a boundary point of a set denoted by $T(G, t_1)$ where $T(G, t)$ is the image of G under the linear mapping corresponding to $\Phi^{-1}(t)$, i.e., $y \in T(G, t)$ if there exists an $x \in G$ such that $y = \Phi^{-1}(t)\, x$. The vector $x^0 - \Phi^{-1}(t_1)\, w(\eta, t_1) + z(t_1, \eta)$ is readily determined as well as its projection on the vector η. Figure 2 depicts the situation when Eq. (5) holds, that is, for the given values of η and x^0, $f(t_2, \eta; x^0) = 0$.

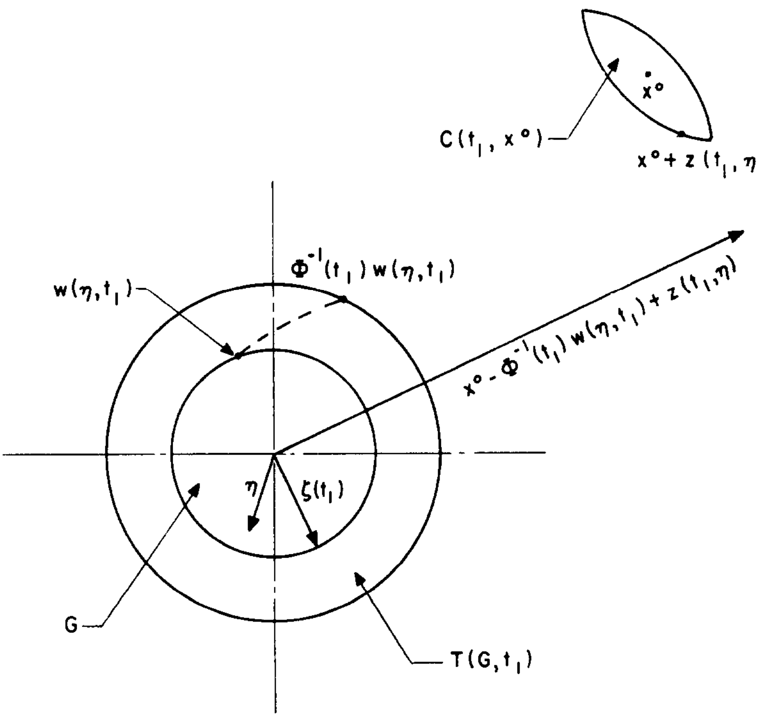

FIG. 1. Geometrical content of Eq. (6)

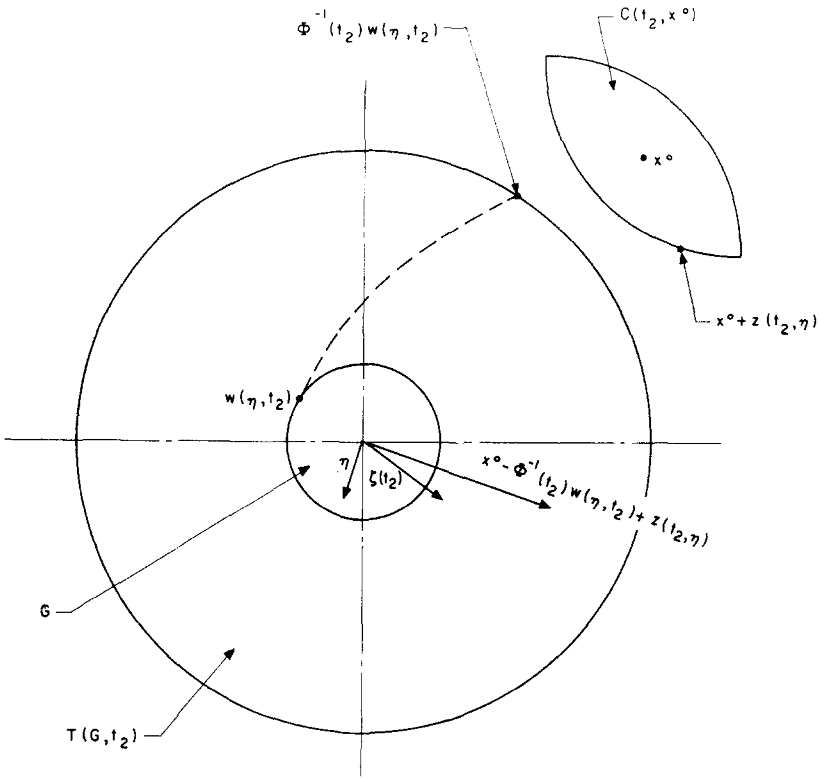

FIG. 2. Geometrical content of Eq. (5)

Now with η fixed $f(t, \eta; x^0)$ will be considered as a function of t. It is evident that

$$\frac{\partial}{\partial t}(\eta \cdot x^0) = 0$$

and

$$\frac{\partial}{\partial t}[\eta \cdot z(t, \eta)] = \eta \cdot \Phi^{-1}(t)\, B(t)\, u(\eta, t) = \zeta(t) \cdot B(t)\, u(\eta, t).$$

Also

$$\frac{\partial}{\partial t}[\eta \cdot \Phi^{-1}(t)\, w(\eta, t)] = \frac{\partial}{\partial t}[\zeta(t) \cdot w(\eta, t)]$$

$$= \dot{\zeta}(t) \cdot w(\eta, t) + \zeta(t) \cdot \frac{\partial}{\partial t} w(\eta, t)$$

if the last term exists. But $\zeta(t) \cdot (\partial/\partial t)\, w(\eta, t)$ is zero since $w(\eta, t)$ is a point on the boundary of G at which $\zeta(t)$ is normal to a support of G and thus

$(\partial/\partial t)\, w(\eta, t)$ is zero or is in the support of G which is normal to $\zeta(t)$. The first term is $-\zeta(t) \cdot A(t)\, w(\eta, t)$ so that

$$\frac{\partial}{\partial t} f(t, \eta; x^0) = \zeta(t) \cdot [A(t)\, w(\eta, t) + B(t)\, u(\eta, t)] \tag{7}$$

which by the definition of $w(\eta, t)$ is evidently nonnegative wherever $w(\eta, t)$ is defined. Thus for any fixed η, $f(t, \eta; x^0)$ is a monotone nondecreasing differentiable function of t wherever it is defined. Suppose (5) has a solution $t(\eta; x^0)$. Then $t(\eta; x^0) \leqslant 0$ if $\eta \cdot [x^0 - w(\eta, 0)] = f(0, \eta; x^0) \geqslant 0$ and hence restrict η so that $\eta \cdot [x^0 - w(\eta, 0)] < 0$.

Now it will be assumed that there exists an optimal control, i.e., there exists a set $H_0(x^0)$ and a $t_1 > 0$ such that (4) is satisfied for $T = t_1$ and $\eta_0 \in H_0(x^0)$. Then

$$z(t_1, \eta^0) = -x^0 + \Phi^{-1}(t_1)\, w(\eta^0, t_1) \in C(t_1)$$

for $w(\eta^0, t_1)$ properly chosen $(w(\eta_0, t_1) = x(t_1))$. If $\eta \notin H_0(x^0)$ then

$$\eta \cdot z(t_1, \eta) > \eta \cdot z(t_1, \eta^0) = -\eta \cdot x^0 + \eta \cdot \Phi^{-1}(t_1)\, w(\eta^0, t_1).$$

Therefore

$$\begin{aligned} f(t_1, \eta; x^0) &= \eta \cdot [z(t_1, \eta) + x^0 - \Phi^{-1}(t_1)\, w(\eta, t_1)] \\ &> \eta \cdot \Phi^{-1}(t_1)\, [w(\eta^0, t_1) - w(\eta, t_1)]. \end{aligned}$$

But

$$\eta \cdot \Phi^{-1}(t_1)\, [w(\eta^0, t_1) - w(\eta, t_1)] = \zeta(t_1) \cdot [w(\eta^0, t_1) - w(\eta, t_1)] \geqslant 0.$$

Hence, $f(t_1, \eta; x^0) > 0$ if it is defined. This property then allows a steepest ascent procedure to be used in order to maximize the function $T(\eta)$, which is defined as the smallest root of (5), over the set of η's for which $f(t, \eta; x^0)$ is defined on $[0, t_1]$.

Let $F(\eta, x^0)$ be defined on this set of η's by

$$F(\eta, x^0) = f(T(\eta), \eta; x^0) \tag{8}$$

Then $F(\eta, x^0) = 0$ and hence

$$\begin{aligned} 0 = \nabla F(\eta, x^0) &= x^0 + z(T(\eta), \eta) - \Phi^{-1}(T(\eta))\, w(\eta, T(\eta)) \\ &+ \{\eta \cdot \Phi^{-1}(T(\eta))\, [A(T(\eta))\, w(\eta, T(\eta)) + B(T(\eta))\, u(\eta, T(\eta))]\}\, \nabla T(\eta). \end{aligned} \tag{9}$$

To implement an ascent procedure it is desired to vary η in such a manner that $T(\eta)$ increases monotonically whenever $T(\eta)$ is not a maximum. This can be accomplished by introducing a real parameter τ and choosing η as a function of τ governed by the differential equation

$$\frac{d\eta}{d\tau} = K(\eta)\, g(\eta) \tag{10}$$

since then

$$\frac{d}{d\tau}\, T(\eta(\tau)) = \| g(\eta) \|^2$$

where

$$K(\eta) = \eta \cdot \{\Phi^{-1}(T(\eta))\, [A(T(\eta))\, w(\eta,\, T(\eta)) + B(T(\eta))\, u(\eta,\, T(\eta))]\} \geqslant 0$$

and

$$-\, g(\eta) = x^0 + z(T(\eta), \eta) - \Phi^{-1}(T(\eta))\, w(\eta,\, T(\eta)) \neq 0$$

if $\eta \notin H_0(x^0)$. Equation (9) implies that $K(\eta) = 0$ only if $g(\eta) = 0$.

Assume now that G satisfies condition (A): For any y in G and any $t_0 > 0$, there exists a control $u(t; y, t_0)$, satisfying $| u_i(t; y, t_0) | \leqslant 1$, $t > t_0$ such that the solution $x(t; y, t_0)$ of (1) with $x(t_0) = y$ and $u(t) = u(t; y, t_0)$ remains within G for all $t > t_0$. Then $w(\eta, t)$ is defined for every η and all $t > 0$. Hence $f(t_1, \eta; x^0)$ is defined and so $T(\eta)$ is defined satisfying $0 < T(\eta) \leqslant t_1$ for η such that $\eta \cdot [x^0 - w(\eta, 0)] < 0$.

Let

$$H(x^0) = \{\eta : \eta \cdot [x^0 - w(\eta, 0)] < 0\}$$

and consider the case when $g(\eta)$ is continuous on $H(x^0)$. Then every solution of (10) with $\eta(0) \in H(x^0)$ satisfies $\eta(\tau) \in H(x^0)$ for all $\tau > 0$, because

$$\eta \cdot \frac{d\eta}{d\tau} = -\, K(\eta)\, F(\eta, x^0) = 0 \qquad \text{and} \qquad T(\eta(\tau)) \geqslant T(\eta(0)) > 0$$

which implies

$$\eta(\tau) \cdot [x^0 - w(\eta(\tau), 0)] \neq 0$$

for all $\tau > 0$. Thus

$$\eta(\tau) \cdot [x^0 - w(\eta(\tau), 0)] < 0$$

for all $\tau > 0$ since $\eta(\tau) \cdot [x^0 - w(\eta(\tau), 0)]$ is continuous and

$$\eta(0) \cdot [x^0 - w(\eta(0), 0)] < 0.$$

Note that $T(\eta)$ is defined on $H(x^0)$, has a maximum for $\eta \in H_0(x^0)$, and has no other extrema. Note that if G is strictly convex then $g(\eta)$ is continuous.

It will be shown that $T(\eta(\tau)) \to t_1$ as $\tau \to \infty$. Suppose the contrary, i.e., there exists a $\delta > 0$ such that $T(\eta(\tau)) \leqslant t_1 - \delta$ for $\tau \geqslant 0$. $T(\eta(\tau)) \geqslant T(\eta(0))$ for $\tau \geqslant 0$ since $dT/d\tau > 0$. Let $H_1 \subset H(x^0)$ be defined by

$$H_1 = \{\eta \in H(x^0) : T(\eta(0)) \leqslant T(\eta) \leqslant t_1 - \delta\}.$$

Then on H_1 there exists an $\epsilon > 0$ such that $\| g(\eta) \|^2 > \epsilon$. Thus $\eta(\tau) \in H_1$ for $\tau \geqslant 0$ and $(d/d\tau)\, T(\eta(\tau)) > \epsilon$ which implies $T(\eta(\tau)) \to \infty$ as $\tau \to \infty$ which is a contradiction.

Remarks

1. Condition (A) is equivalent to (B): Every direction is admitted by the transversality condition, i.e., $w(\eta, 0)$ is defined for all η.

2. Suppose $G = \{y\}$ and condition (A) is satisfied, i.e., there exists a control $\bar{u}(t)$ such that $| \bar{u}_i(t) | \leqslant 1$ and $A(t)\, y + B(t)\, \bar{u}(t) = 0$. Then $w(\eta, t) = y$ and the result obtained for continuous $g(\eta)$ holds. Also it should be noted that no difficulty is encountered if $y = y(t)$.

References

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